

Boolean Differential Operators

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Abstract

We consider four combinatorial interpretations for the algebra of Boolean differential operators. We show that each interpretation yields an explicit matrix representation for Boolean differential operators.

1 Introduction

Fix $n \in \mathbb{N}$. A Boolean function, also known as a truth function, with n -arguments is a map

$$f : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2,$$

where $\mathbb{Z}_2 = \{0, 1\}$ is the field with two elements. The \mathbb{Z}_2 -algebra BF_n of Boolean functions on n -arguments, given by pointwise sum and multiplication, is isomorphic to the Boolean algebra $\text{PP}[n]$ of sets of subsets of $[n] = \{1, \dots, n\}$. Indeed, we identify $a \in \mathbb{Z}_2^n$ with an element of $\text{P}[n]$ via the characteristic function, and we identify a map $\mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ with a subset of $\text{P}[n]$, again with the help of characteristic functions. The sum and product of Boolean functions correspond, respectively, with the symmetric difference and the intersection of subsets of $\text{P}[n]$. The isomorphism $\text{BF}_n \simeq \text{PP}[n]$ establishes the link between classical propositional logic and set theory [1].

The partial derivative $\partial_i f : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ of the Boolean function $f : \mathbb{Z}_2^n \longrightarrow \mathbb{Z}_2$ is given [3] by

$$\partial_i f(a) = f(a + e_i) + f(a),$$

where the vectors e_i form the canonical basis of \mathbb{Z}_2^n . We define, in analogy with the classical case, the \mathbb{Z}_2 -algebra of Boolean differential operators BDO_n as the subalgebra of $\text{End}_{\mathbb{Z}_2}(\text{BF}_n)$ generated by the operators of multiplication by Boolean functions, and the partial derivatives operators ∂_i . It turns out that $\text{BDO}_n = \text{End}_{\mathbb{Z}_2}(\text{BF}_n)$, i.e. any linear operator from BF_n to itself is actually given by a Boolean differential operator [2].

It is desirable to have a set theoretical interpretation for the algebras BDO_n , extending the usual interpretation of BF_n as the Boolean algebra $\text{PP}[n]$. By dimension counting BDO_n is isomorphic to the \mathbb{Z}_2 -algebra $\text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$ of square matrices of size 2^n with 0-1 entries. Note

that $M_{2^n \times 2^n}(\mathbb{Z}_2)$ may be further identified, via characteristic functions, with $P(P[n] \times P[n])$ the set of subsets of $P[n] \times P[n]$, or equivalently, with the set $DG_{P[n]}$ of simple directed graphs with vertex set $P[n]$. A matrix $A \in M_{2^n \times 2^n}(\mathbb{Z}_2)$, i.e. an element $A \in P(P[n] \times P[n])$, is regarded as a directed graph by drawing an edge from vertex b to vertex a if and only if $(a, b) \in A$. The sum of digraphs in $DG_{P[n]}$ is given by the symmetric difference; the product AB of digraphs $A, B \in DG_{P[n]}$ is such that the pair $(a, b) \in AB$ if and only if there is an odd number of sets $c \in P[n]$ such that $(a, c) \in A$ and $(c, b) \in B$.

In order to make explicit the identification $BDO_n \simeq M_{2^n \times 2^n}(\mathbb{Z}_2)$ a choice of bases for BDO_n and BF_n must be made. In this work we fix a particular basis for BF_n , indeed we always consider the basis $\{ m^a \mid a \in P[n] \}$, where the Boolean function m^a is given on $b \in P[n]$ by:

$$m^a(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we consider four different bases for BDO_n . Having various presentations for BDO_n available is desirable, just as in the case of Boolean functions, it is convenient to generate truth functions by several types of logical connectives. Each choice of basis yields a different product (all of them isomorphic) on set $DG_{P[n]}$ of directed graphs with vertex set $P[n]$. Such products, denoted respectively by $\star, \circ, \bullet, *$, were introduced in [2]. It turns out that the product \star is the easiest to handle, we discuss some of its properties and describe an explicit isomorphism with $M_{2^n \times 2^n}(\mathbb{Z}_2)$ in Section 2. In the remaining Sections, we present explicit isomorphisms between the products $\circ, \bullet, *$, and product of square matrices with entries in \mathbb{Z}_2 of size 2^n .

2 MS Basis and the \star Product

As mention in the introduction we consider four different bases for the algebra Boolean differential operators BDO_n . In this section we consider the MS-basis $\{ m^a s^b \mid a, b \in P[n] \}$, where the Boolean functions m^a were described in the introduction, and the shift operators $s^b : BF_n \rightarrow BF_n$ are given by

$$s^b = \prod_{i \in b} s_i \quad \text{where} \quad s_i f(a) = f(a + e_i).$$

Note that $s_i = \partial_i + 1$, thus one can move back and forward from shift operators $s^b = \prod_{i \in b} s_i$ to partial derivatives operators $\partial^b = \prod_{i \in b} \partial_i$. Indeed, it is easy to see, with the help of the Möbius inversion formula that

$$\partial^b = \sum_{a \subseteq b} s^a \quad \text{and} \quad s^b = \sum_{a \subseteq b} \partial^a.$$

Consider the identifications

$$DG_{P[n]} \simeq \text{Map}(P[n] \times P[n], \mathbb{Z}_2) \simeq BDO_n,$$

where the first identification is given by the characteristic functions and is freely used without change of notation; the second non-canonical identification is obtained by regarding a directed graph $A \in \text{DG}_{\mathbb{P}[n]}$ as Boolean differential operator as follows

$$A \leftrightarrow \sum_{(a,b) \in A} m^a s^b \leftrightarrow \sum_{a,b \in \mathbb{P}[n]} A(a,b) m^a s^b.$$

With this identification the composition product on BDO_n induces a product on $\text{DG}_{\mathbb{P}[n]}$, which we denoted by \star , defined for $A, B \in \text{DG}_{\mathbb{P}[n]}$ by the following equivalent identities, see [2]:

$$A \star B(a_1, a_2) = \sum_{b \in \mathbb{P}[n]} A(a_1, b) B(a_1 + b, a_2 + b).$$

$$A \star B = \{ (a_1, a_2) \in \mathbb{P}[n] \times \mathbb{P}[n] \mid O\{b \in \mathbb{P}[n] \mid (a_1, b) \in A, (a_1 + b, a_2 + b) \in B\} \},$$

where the notation OC means that the finite set C has odd cardinality.

Lemma 1. Let $a, b, c, d \in \mathbb{P}[n]$, then we have that:

$$\{(a, b)\} \star \{(c, d)\} = \begin{cases} \{(a, b + d)\} & \text{if } a = b + c, \\ \emptyset & \text{if } a \neq b + c. \end{cases}$$

Proof. Let $(a_1, a_2) \in \{(a, b)\} \star \{(c, d)\}$, then there is an odd number of sets $e \in \mathbb{P}[n]$ such that

$$(a_1, e) = (a, b) \quad \text{and} \quad (a_1 + e, a_2 + e) = (c, d).$$

Thus $a_1 = a, e = b, a = b + c, a_2 = b + d$. So, there is no such e if $a \neq b + c$, and in the later case $e = b$ and $(a_1, a_2) = (a, b + d)$. \square

Proposition 2. Let $A, B \in \text{DG}_{\mathbb{P}[n]}$ be given by

$$A = \sum_{b \in \mathbb{P}[n]} A_b \times \{b\} \quad \text{and} \quad B = \sum_{c \in \mathbb{P}[n]} B_c \times \{c\}.$$

Then we have that:

$$A \star B = \sum_{b, c \in \mathbb{P}[n]} (A_b \cap (B_c + b)) \times \{b + c\}.$$

Proof.

$$A \star B = \sum_{b, c \in \mathbb{P}[n]} (A_b \times \{b\}) \star (B_c \times \{c\}) = \sum_{b, c \in \mathbb{P}[n]} (A_b \cap (B_c + b)) \times \{b + c\}.$$

Suppose $(a_1, a_2) \in (A_b \times \{b\}) \star (B_c \times \{c\})$, then there is an odd number of sets $e \in \mathbb{P}[n]$ such that $(a_1, e) \in A_b \times \{b\}$ and $(a_1 + e, a_2 + e) \in B_c \times \{c\}$. Clearly, the only possible set e is $e = b$, and furthermore $a_1 \in A_b, a_2 = b + c$, and $a_1 \in B_c + b$. \square

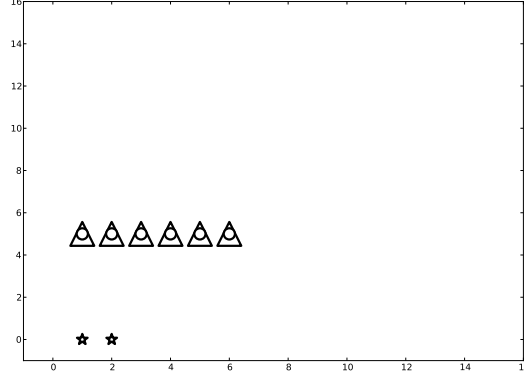


Figure 1: $(A_{\{1,2\}} \times \{1,2\}) \star (A_{\{1,2\}} \times \{1,2\})$

Example 3. Let $A_{\{1,2\}} = \{\{1,3\}, \{3\}, \{4\}, \{2\}, \{1,2\}, \{1\}\} \in \text{PP}[4]$, then

$$(A_{\{1,2\}} \times \{1,2\}) \star (A_{\{1,2\}} \times \{1,2\}) = \{\{1\}, \{2\}\} \times \{\emptyset\}.$$

Figure 1 shows the product in Example 3. In all Figures in this work we use the following the conventions. We draw a subset of $\text{P}[n] \times \text{P}[n]$ as a subset of the plane using the one-to-one correspondence between subsets of $\text{P}[n] = \mathbb{Z}_2^n$ and natural numbers in the interval $[0, 2^n - 1]$ given by cardinality and lexicographic order within a given cardinality. For example $\text{P}[2]$ and $[0, 2^2 - 1]$ are in correspondence as follows $\emptyset \rightarrow 0$, $\{1\} \rightarrow 1$, $\{2\} \rightarrow 2$, $\{1,2\} \rightarrow 3$. When we consider a product, the elements of first factor are drawn as triangles; the elements of the second factor are drawn as circles; and the elements of the product are drawn as stars. For matrices in $\text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$ we always refer to the cardinality-lexicographic order on $\mathbb{Z}_2^n = \text{P}[n]$.

Proposition 4. Let $A, B \in \text{DG}_{\text{P}[n]}$ and write

$$A = \sum_{b \in \text{P}[n]} A_b \times \{b\} \quad \text{and} \quad B = \sum_{c \in \text{P}[n]} \{c\} \times B_c.$$

Then we have that:

$$A \star B = \sum_{b, c \in \text{P}[n]} (A_b \cap (A_c + b)) \times \{b + c\}.$$

Proof.

$$A \star B = \sum_{a, b \in \text{P}[n]} (A_b \times \{b\}) \star (\{a\} \times B_a) = \{a + b\} \times (B_a + b),$$

if $a + b \in A_b$ and zero otherwise. Suppose $(a_1, a_2) \in (A_b \times \{b\}) \star (\{a\} \times B_a)$, then there is a odd number of sets $e \in \text{P}[n]$ such that $(a_1, e) \in A_b \times \{b\}$ and $(a_1 + e, a_2 + e) \in \{a\} \times B_a$. Clearly, the only possibility for e is $e = b$, and furthermore $a_1 = a + b \in A_b$, $a_2 \in B_a + b$. \square

Example 5. Let $B_{\{1,3\}}$ and $A_{\{1,2\}}$ in $\text{PP}[4]$ be given by

$$B_{\{1,3\}} = \{\{1,2\}, \{4\}, \{1\}, \{2\}, \emptyset, \{3\}\} \quad \text{and} \quad A_{\{1,2\}} = \{\{1,3\}, \{3\}, \{4\}, \{2\}, \{1,2\}, \{1\}\}.$$

Then we have, see Figure 2, that:

$$(\{1,3\} \times B_{\{1,3\}}) \star (A_{\{1,2\}} \times \{1,2\}) = (\{\{1,3\}\} \times \{\{1,2\}, \{1,2,3\}, \{2\}\}).$$

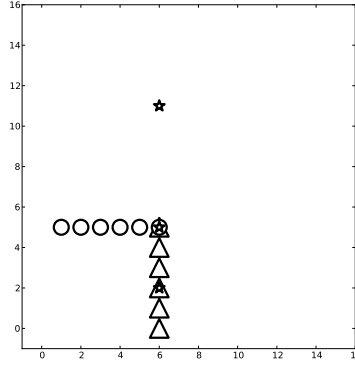


Figure 2: $(\{1,3\} \times B_{\{1,3\}}) \star (A_{\{1,2\}} \times \{1,2\})$.

Proposition 6. Let $A, B \in \text{DGP}[n]$ and write

$$A = \sum_{c \in \text{P}[n]} \{c\} \times \{A_c\} \quad \text{and} \quad B = \sum_{d \in \text{P}[n]} \{d\} \times \{B_d\}.$$

Then we have that:

$$A \star B = \sum_{c+d \in A_c} \{c\} \times (B_d + c + d).$$

Proof. We have that

$$A \star B = \sum_{c,d \in \text{P}[n]} (\{c\} \times A_c) \star (\{d\} \times B_d).$$

A pair $(a_1, a_2) \in \text{P}[n] \times \text{P}[n]$ belongs to $(\{c\} \times A_c) \star (\{d\} \times B_d)$ if there is an odd number of sets $e \in \text{P}[n]$ such that $(a_1, e) \in \{c\} \times A_c$ and $(a_1 + e, a_2 + e) \in \{d\} \times B_d$. Thus we have that $a_1 = c$, $e \in A_c$, $e = c + d$, $a_2 \in B_d + c + d$. The result follows. □

Example 7. Let $B_{\{1,3\}}$ and A_\emptyset in $\text{PP}[4]$ be given by

$$A_\emptyset = \{\{3\}, \{4\}, \{1,4\}, \{2\}, \{1,2\}, \{1,3\}\} \quad \text{and} \quad B_{\{1,3\}} = \{\{1,2\}, \{4\}, \{1\}, \{2\}, \emptyset, \{3\}\}.$$

Then, see Figure 3, we have that:

$$(\{\emptyset\} \times A_\emptyset) \star (\{\{1, 3\}\} \times B_{\{1,3\}}) = (\{\emptyset\} \times \{\{3\}, \{1, 3, 4\}, \{1\}, \{1, 2, 3\}, \{2, 3\}, \{1, 3\}\}).$$

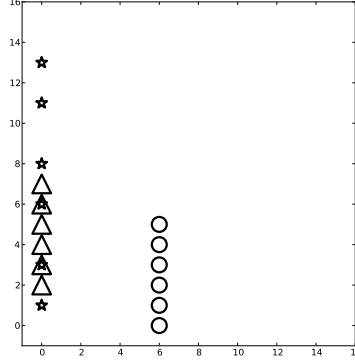


Figure 3: $(\{\emptyset\} \times A_\emptyset) \star (\{\{1, 3\}\} \times B_{\{1,3\}})$

Example 8. Let

$$\begin{aligned} A &= \{(\{1, 3\}, \{1, 2\}), (\{1, 3\}, \{4\}), (\{4\}, \{1, 3\}), (\{1, 2\}, \{1, 2\}), (\{1, 3\}, \{1, 3\}), \\ &\quad (\{4\}, \{1, 2\}), (\{4\}, \{4\}), (\{1, 2\}, \{4\}), (\{1, 2\}, \{1, 3\})\}, \quad \text{and} \\ B &= \{(\{1, 3\}, \{2, 3\}), (\{1, 4\}, \{1, 4\}), (\{1, 4\}, \{2, 3\}), (\{2, 3\}, \{1, 3\}), (\{1, 3\}, \{1, 3\}), \\ &\quad (\{1, 4\}, \{1, 3\}), (\{1, 3\}, \{1, 4\}), (\{2, 3\}, \{1, 4\}), (\{2, 3\}, \{2, 3\})\}. \end{aligned}$$

The product $A \star B$, see Figure 4, is given by

$$\{(\{1, 3\}, \{2, 3\}), (\{1, 3\}, \{1, 3\}), (\{1, 2\}, \{3, 4\}), (\{1, 3\}, \{2, 4\}), (\{1, 2\}, \{1, 2\}), (\{1, 2\}, \emptyset)\}.$$

Theorem 9. The algebra $(\text{DG}_{\text{P}[n]}, \star)$ is isomorphic to $\text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$ via the map

$$M : \text{P}(\text{P}[n] \times \text{P}[n]) \longrightarrow \text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$$

sending $A \in \text{DG}_{\text{P}[n]}$ to the matrix $M(A) \in \text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$ given by $M(A)_{a,b} = A(a, a+b)$. The inverse map

$$\text{M}_{2^n \times 2^n}(\mathbb{Z}_2) \longrightarrow \text{DG}_{\text{P}[n]}$$

sends a matrix $M \in \text{M}_{2^n \times 2^n}(\mathbb{Z}_2)$ to the directed graph $A_M \in \text{DG}_{\text{P}[n]}$ with characteristic function given by $A_M(a, b) = M_{a, a+b}$.

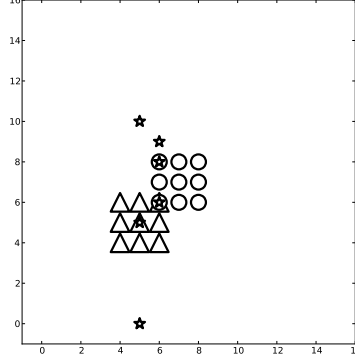


Figure 4: Product $A \star B$ of sets defined in Example 8 .

Proof. We have $m^c s^d(m^b) = \delta(c, b + d)m^c$ and therefore $M(m^c s^d)_{a,b} = \delta(c, a)\delta(a, b + d)$. Thus

$$M(A)_{a,b} = \sum_{c,d \in P[n]} A(c, d)M(m^c s^d)_{a,b} = \sum_{c,d \in P[n]} A(c, d)\delta(c, a)\delta(a, b + d) = A(a, a + b).$$

Also we have that:

$$M(M(A))_{a,b} = M(A)(a, a + b) = A(a, a + a + b) = A(a, b).$$

□

Example 10. The multiplication table of $(\text{DG}_{P[1]}, \star)$ is given in Table 6.

Example 11. Consider the Jordan-like matrices $M_{2^n \times 2^n}(\mathbb{Z}_2)$ having ones on the principal diagonal and on the diagonal directly above the principal. The associated Boolean differential operators in the MS-basis, for $n \in [4]$, are shown in the following table.

n	Operator
1	$m^\emptyset s^{\{1\}} + 1$
2	$m^{\{2\}} s^{\{1\}} + m^\emptyset s^{\{1\}} + m^{\{1\}} s^{\{1,2\}} + 1$
3	$m^{\{1,3\}} s^{\{1,2\}} + m^{\{2,3\}} s^{\{1\}} + m^{\{3\}} s^{\{1,2,3\}} + m^{\{1,2\}} s^{\{2,3\}} + m^\emptyset s^{\{1\}} + m^{\{2\}} s^{\{2,3\}} + m^{\{1\}} s^{\{1,2\}} + 1$
4	$m^{\{2,3,4\}} s^{\{1\}} + m^{\{1,2,3\}} s^{\{3,4\}} + m^{\{1,4\}} s^{\{1,2,3,4\}} + m^{\{3\}} s^{\{3,4\}} + m^{\{2,4\}} s^{\{2,3\}} + m^{\{2,3\}} s^{\{3,4\}} + m^{\{1\}} s^{\{1,2\}} + m^{\{1,3\}} s^{\{3,4\}} + m^{\{1,3,4\}} s^{\{1,2\}} + m^{\{4\}} s^{\{1,2,4\}} + m^{\{1,2,4\}} s^{\{2,3\}} + m^\emptyset s^{\{1\}} + m^{\{1,2\}} s^{\{2,3\}} + m^{\{2\}} s^{\{2,3\}} + m^{\{3,4\}} s^{\{1,2,4\}} + 1$

Corolary 12. $|\text{Ker}(A)| = 2^r$ and $|\text{Im}(A)| = 2^{n-r}$, where $r = \text{rank}(M(A))$ is the rank of $M(A)$.

3 $M\partial$ Basis and \circ Product

In this section we consider the $M\partial$ basis on BDO_n , i.e. we regard a directed graph $A \in DG_{P[n]}$ as a Boolean differential operator via the identifications

$$A \leftrightarrow \sum_{a,b \in A} m^a \partial^b \leftrightarrow \sum_{a,b \in P[n]} A(a,b) m^a \partial^b$$

where we recall that for $b \in P[n]$ we set $\partial^b = \prod_{i \in b} \partial_i$. The product on $DG_{P[n]}$ corresponding to the composition product on BDO_n via this identification is denoted by \circ .

The \circ product of sets $A, B \in DG_{P[n]}$ coming from the composition of Boolean operators, see [2], is given by:

$$A \circ B(a, b) = \sum_{\substack{c, d, e \subseteq b \\ b \setminus e \subseteq a + d \subseteq c}} A(a, c) B(d, e).$$

Equivalently, a pair $(a_1, a_2) \in P[n] \times P[n]$ belongs to $A \circ B \in DG_{P[n]}$ if and only if there is an odd number of sets $b \in P[n]$ and $(c_1, c_2) \in B$ such that

$$(a_1, b) \in A, \quad c_2 \subseteq a_2, \quad a_2 \setminus c_2 \subseteq a_1 + c_1 \subseteq b.$$

Note that we can go back and forward from the MS-basis to the $M\partial$ -basis for Boolean differential operators as follows:

$$A = \sum_{a,b \in P[n]} A(a,b) m^a \partial^b = \sum_{a,b \in P[n]} \hat{A}(a,b) m^a s^b,$$

where

$$\hat{A}(a,b) = \sum_{b \subseteq c} A(a,c) \quad \text{and} \quad A(a,b) = \sum_{b \subseteq c} \hat{A}(a,c).$$

Theorem 13. The algebra $(DG_{P[n]}, \circ)$ is isomorphic to $M_{2^n \times 2^n}(\mathbb{Z}_2)$ via the map sending a graph $A \in DG_{P[n]}$ to the matrix $M(A) \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ given by

$$M(A)_{a,b} = \sum_{a+b \subseteq c} A(a,c).$$

The inverse map $M_{2^n \times 2^n}(\mathbb{Z}_2) \longrightarrow DG_{P[n]}$ sends a matrix $M \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ to the directed graph $A_M \in DG_{P[n]}$ with characteristic function given by

$$A_M(a,b) = \sum_{b \subseteq c} M_{a,a+c}.$$

Proof. We have that

$$M(A)_{a,b} = \sum_{c, e \subseteq d} A(c,d) M(m^c s^e)_{a,b} = \sum_{c, e \subseteq d} A(c,d) \delta(c,a) \delta(a,b+e) = \sum_{a+b \subseteq d} A(a,d).$$

We also have for $A \in \text{DG}_{\mathbf{P}[n]}$ that:

$$A_{M(A)}(a, b) = \sum_{b \subseteq c} M(A)_{a, a+c} = \sum_{b \subseteq c} \left(\sum_{a+a+c \subseteq d} A(a, d) \right) = \sum_{b \subseteq c \subseteq d} A(a, d) = A(a, b).$$

□

Corolary 14. $|\text{Ker}(A)| = 2^r$ and $|\text{Im}(A)| = 2^{n-r}$, where $r = \text{rank}(M(A))$ is the rank of $M(A)$.

4 XS Basis and the $*$ Product

For $a \in \mathbf{P}[n]$ consider the Boolean function $x^a \in \text{BF}_n$ given on $b \in \mathbf{P}[n]$ by:

$$x^a(b) = \begin{cases} 1 & \text{if } a \subseteq b, \\ 0 & \text{otherwise.} \end{cases}$$

Note that one can go back and forward from the m^a basis to the x^a basis, with the help of the with the help of the Möbius inversion formula as, as follows:

$$m^a = \sum_{a \subseteq b} x^b \quad \text{and} \quad x^a = \sum_{a \subseteq b} m^b.$$

In this section we regard elements of $\text{DG}_{\mathbf{P}[n]}$ as operators in BDO_n via the identification

$$A \leftrightarrow \sum_{a, b \in A} x^a s^b \leftrightarrow \sum_{a, b \in \mathbf{P}[n]} A(a, b) x^a s^b$$

The product on $\text{DG}_{\mathbf{P}[n]}$ obtained via this identification is denoted by $*$.

The $*$ product of graphs $A, B \in \text{DG}_{\mathbf{P}[n]}$ is given, see [2], by:

$$A * B(a, b) = \sum_{c \subseteq f, d, e} |\{k \subseteq d \cap e \mid c \cup e \setminus k = f\}| A(c, d) B(e, b + d).$$

Equivalently, a pair $(a_1, a_2) \in \mathbf{P}[n] \times \mathbf{P}[n]$ belongs to $A * B \in \text{DG}_{\mathbf{P}[n]}$ if and only if there is an odd number of sets $b, c, d, e \in \mathbf{P}[n]$ such that

$$b \subseteq a_1, \quad e \subseteq c \cap d, \quad b \cup d \setminus e = a_1, \quad (b, c) \in A, \quad (d, a_2 + c) \in B.$$

Note that we can go back and forward from the MS-basis to the XS-basis for differential operators as follows:

$$A = \sum_{a, b \in \mathbf{P}[n]} A(a, b) x^a s^b = \sum_{a, b \in \mathbf{P}[n]} \hat{A}(a, b) m^a s^b,$$

where

$$\hat{A}(a, b) = \sum_{c \subseteq a} A(c, b) \quad \text{and} \quad A(a, b) = \sum_{c \subseteq a} \hat{A}(c, b).$$

Theorem 15. The algebra $(\text{DGP}_{[n]}, *)$ is isomorphic to $M_{2^n \times 2^n}(\mathbb{Z}_2)$ via the map sending a graph $A \in \text{DGP}_{[n]}$ to the matrix $M(A) \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ given by

$$M(A)_{a,b} = \sum_{c \subseteq a} A(c, a+b).$$

The inverse map $M_{2^n \times 2^n}(\mathbb{Z}_2) \longrightarrow \text{DGP}_{[n]}$ sends a matrix $M \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ to the graphs $A_M \in \text{DGP}_{[n]}$ with characteristic function given by

$$A_M(a, b) = \sum_{c \subseteq a} M_{c, b+c}.$$

Proof. We have that

$$M(A)_{a,b} = \sum_{c \subseteq e, d} A(c, d) M(m^e s^d)_{a,b} = \sum_{c \subseteq e, d} A(c, d) \delta(e, a) \delta(e, b+d) = \sum_{c \subseteq a} A(c, a+b).$$

We also have for $A \in \text{DGP}_{[n]}$ that:

$$A_{M(A)}(a, b) = \sum_{c \subseteq a} M(A)_{c, b+c} = \sum_{c \subseteq a} \left(\sum_{d \subseteq c} A(d, b) \right) = \sum_{d \subseteq c \subseteq a} A(d, b) = A(a, b).$$

□

Corolary 16. $|\text{Ker}(A)| = 2^r$ and $|\text{Im}(A)| = 2^{n-r}$, where $r = \text{rank}(M(A))$ is the rank of $M(A)$.

5 $X\partial$ Basis and the \bullet Product

In this Section we regard elements of $\text{DGP}_{[n]}$ as Boolean differential operators via the identification the identification

$$A \leftrightarrow \sum_{a, b \in A} x^a \partial^b \leftrightarrow \sum_{a, b \in \text{P}[n]} A(a, b) x^a \partial^b$$

The product on $\text{DGP}_{[n]}$ obtained via this identification is denoted by \bullet .

The \bullet -product of sets $A, B \in \text{DGP}_{[n]}$ is given by, see [2]:

$$A \bullet B(a, b) = \sum_{c \subseteq a, d, e, f \subseteq b} |\{k_1 \subseteq k_2 \subseteq d \cap e \mid c \cup (e \setminus k_2) = a, d \setminus k_1 = b \setminus f\}| A(c, d) B(e, f).$$

Equivalently, a pair $(a_1, a_2) \in \text{P}[n] \times \text{P}[n]$ belongs to $A \bullet B \in \text{DGP}_{[n]}$ if and only if there is an odd number of sets $(b_1, b_2) \in A, (c_1, c_2) \in B, k_1 \subseteq k_2 \subseteq [n]$ such that

$$b_1 \subseteq a_1, \quad c_1 \subseteq a_2, \quad k_2 \subseteq b_2 \cap c_1, \quad b_1 \cup (c_1 \setminus k_2) = a_1, \quad b_2 \setminus k_1 = a_2 \setminus c_2.$$

Note that we can go back and forward from the MS-basis to the XS-basis for differential operators as follows:

$$A = \sum_{a,b \in P[n]} A(a,b) x^a \partial^b = \sum_{a,b \in P[n]} \hat{A}(a,b) m^a s^b,$$

where

$$\hat{A}(a,b) = \sum_{e \subseteq a, b \subseteq f} A(e,f) \quad \text{and} \quad A(a,b) = \sum_{e \subseteq a, b \subseteq f} \hat{A}(e,f).$$

Theorem 17. The algebra $(\text{DGP}[n], \bullet)$ is isomorphic to $M_{2^n \times 2^n}(\mathbb{Z}_2)$ via the map sending the graph $A \in \text{DGP}[n]$ to the matrix $M(A) \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ given by

$$M(A)_{a,b} = \sum_{c \subseteq a, a+b \subseteq d} A(c,d).$$

The inverse map $M_{2^n \times 2^n}(\mathbb{Z}_2) \longrightarrow P(P[n] \times P[n])$ sends the matrix $M \in M_{2^n \times 2^n}(\mathbb{Z}_2)$ to the graph $A_M \in \text{DGP}[n]$ with characteristic function given by

$$A_M(a,b) = \sum_{c \subseteq a, b \subseteq d} A(c, c+d).$$

Proof. Since $x^a = \sum_{e \subseteq a} m^e$ and $\partial^b = \sum_{f \subseteq b} s^f$, we have that

$$M(A)_{a,b} = \sum_{c \subseteq e, f \subseteq d} A(c,d) M(m^e s^f)_{a,b} = \sum_{c \subseteq e, d} A(c,d) \delta(e,a) \delta(e, f+b) = \sum_{c \subseteq a, a+b \subseteq d} A(c,d).$$

Furthermore, we have that:

$$\begin{aligned} A_{M(A)}(a,b) &= \sum_{c \subseteq a, b \subseteq d} M(A)(c, c+d) = \\ &= \sum_{e \subseteq c, c+c+d \subseteq f} A(e,f) = \sum_{e \subseteq c \subseteq a, b \subseteq d \subseteq f} A(e,f) = A(a,b). \end{aligned}$$

□

Corolary 18. $|\text{Ker}(A)| = 2^r$ and $|\text{Im}(A)| = 2^{n-r}$, where $r = \text{rank}(M(A))$ is the rank of $M(A)$.

6 Final Comments

In this work we introduced four combinatorial interpretations for the composition of Boolean differential operators, and provided a matrix representation for each of these interpretations. Together with the operation of symmetric difference we have thus provided several combinatorial interpretations for the algebra of square matrices of size 2^n with coefficients in \mathbb{Z}_2 . Therefore our work provides set theoretical interpretations for Boolean differential operators. It would be nice to find as well logical interpretations, i.e. some sort of non-commutative logic where Boolean differential operators play the role played by Boolean functions in classical propositional logic. Partial results along this line are developed in [2], where a couple of explicit presentations by generators and relations of the algebra of Boolean differential operators are provided.

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